

Nonlinear random vibration using updated tail equivalent linearization method

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Abstract Tail equivalent linearization method is based on first order reliability method, which obtains an equivalent linear system for the considered nonlinear problem with equal tail probability related to a specified threshold and time. This method has been applied only to nonlinear non-degrading single- and multi-degrees of freedom shear beam two-dimensional models and three-dimensional one-story rigid diaphragm supported by frames with in-plane uni-axial stiffness which is subjected to independent random excitation along the structural axes. To use TELM for more practical problems it is required to extend this method to cover more realistic material and excitation characteristics. In this paper, some of these developments have been presented. Application of TELM for bi-directional excitation with bi-axial material subjected to different incidence angles of excitation and using TELM for degrading material which has been presented in the previous works of the authors have been reviewed briefly. In addition a new method for defining rotational dependent component of earthquake excitation in terms of independent translational components in the standard normal random variable space is proposed, and TELM has been used for this kind of excitation. Three examples related to these extensions have been presented; the comparison of the TELM results with Monte-Carlo simulation results shows good agreement.

Keywords Nonlinear random vibration · Reliability · Linearization · Rotational component

Introduction

Most structures under extreme dynamic loads resulting from natural hazards with low probability exhibit nonlinear behavior. Accurate prediction of variations of such loads is impossible and therefore usually these loads are modeled as random processes. Thus nonlinear random vibration methods are the best methods in the analysis of the structures under severe loads associated with natural hazards. Random vibration for linear structures uses the superposition principle. However, this advantage is not applicable for nonlinear systems, but there are ways to transform a nonlinear system to an equivalent linear system that can benefit from this privilege. In the conventional method i.e. equivalent linearization method (ELM) which is widely used because of its simplicity and applicability to different systems the equivalent system is selected by minimizing the mean-square error between the responses of the nonlinear and the linear systems based on the assumption of Gaussian response for the nonlinear system. Since the Gaussian assumption is not valid for high nonlinear systems, although the accuracy of the method is good in estimating the mean-square response, the probability distribution can be far from correct, particularly in the tail region. Thus estimates of response statistics such as crossing rates and first-passage probability, issues of which are of particular interest in reliability analysis, can be grossly inaccurate at high thresholds.

To overcome the shortcomings of the conventional ELM, Fujimura and Der Kiureghian (2007) presented tail equivalent linearization method (TELM) which uses the advantages of first order reliability method (FORM). In this method stochastic excitation is discretized and represented in terms of a finite set of standard normal random variables. Based on this representation of excitation, the limit state surface for

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desired response at a specified time instant can be stated in terms of these variables. In TELM the nonlinear limit state of the specified response threshold and time is linearized at the nearest point to the origin. Based on the rotational symmetry and exponential decaying of the standard normal probability density function, this point which is called design point in FORM has the maximum likelihood among all points on the limit state surface and has the most contribution in the probability of failure. Tail equivalent linear system (TELS) is defined based on the linearized limit state surface, because its tail probability is equal to the tail probability of the nonlinear system. This definition is accomplished by unit impulse response functions (IRFs) for nonlinear system for each direction of excitation. These IRFs can be used to obtain the statistical properties of the nonlinear system by linear random vibration methods.

This method could predict probability density function (PDF), cumulative distribution function (CDF), crossing rate and first-passage probability with good accuracy. Furthermore, since the linear system with equivalent tail is dependent on the specified threshold the method is capable to predict the non-Gaussian distribution of the nonlinear response. In addition, due to the invariance of TELS on the scale of the excitation estimates for a sequence of scaled excitations (fragility analysis) can be performed with a single determination of the TELS (Fujimura and Der Kiureghian 2007; Der Kiureghian and Fujimura 2009).

TELM has been applied to single- and multi-degrees of freedom 2D shear beam frames and stick-like models by Fujimura and Der Kiureghian (2007) and Der Kiureghian and Fujimura (2009); and to 3D structures with rigid diaphragm subjected to independent bi-directional excitation along the structural axes for uni-axial Bouc–Wen material by Broccardo and Der Kiureghian (2012).

To apply the TELM to more realistic problems, following points need to be considered:

- Bi-axial material behavior for the bi-directional excitation;
- Materials show degradation behavior. This point has not been considered in the previous studies;
- The independent components of excitation are not along the major axes of structure;
- The rotational component of excitation (especially for earthquake excitation). Considering that it is not possible to define this component as a statistically independent component from horizontal components, a new method for simulating this component in terms of translational components of earthquake has been presented.

In a new study, the authors extended TELM to bi-axial Bouc–Wen material subjected to bi-directional excitation along or not along the structural axes in Raoofi and Ghafory-Ashtiany (2013a) and Ghafory-Ashtiany and Raoofi

(2013). Furthermore, the authors have developed modified TELM method for considering degradation of materials for un-axial Bouc–Wen model in 2D and 3D arrangements Raoofi and Ghafory-Ashtiany (2013b). In this paper, in addition to reviewing these findings, a new method for defining the rotational component of earthquake excitation has been presented, and TELM method has been applied with considering this component of excitation.

A brief review on TELM

The first step in the application of FORM or TELM is temporal discretizing of the excitation in terms of standard normal random variables. It is assumed that the components of the multi-directional excitation are statistically independent and can be stated as follows (Rezaeian and Der Kiureghian 2011):

$$f_j(t) = \sum_{i=1}^n u_j^{(i)} s_j^{(i)}(t) = s_j(t) \mathbf{u}_j \quad j = 1, \dots, m \quad (1)$$

is the component of the base excitation in the j th direction at discrete time point $t = t_0, t_1, \dots, t_i, \dots, t_n$ where $t_i = i \cdot \Delta t$. In this representation of excitation $\mathbf{u}_j = \{u_j^{(1)}, u_j^{(2)}, \dots, u_j^{(n)}\}^T$ and $s_j(t) = \{s_j^{(1)}(t), s_j^{(2)}(t), \dots, s_j^{(n)}(t)\}^T$ are vectors of standard normal random variables and deterministic basis functions in the $j = 1, \dots, m$ directions, respectively. Thus $\mathbf{u} = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m]^T$ represents the uncertainty of the excitation and, therefore, is a vector which represents the uncertainty of the problem with $m \times n$ elements. In this paper $s_j(t)$ vectors are calculated based on the presented method in Fujimura and Der Kiureghian (2007) and Rezaeian and Der Kiureghian (2011). For a linear system using superposition role, the desired response $\chi(t)$ which is affected by the multi-component base excitation can be written as follows:

$$\chi(t) = \sum_{j=1}^m \int_0^t f_j(\tau) h_j(t - \tau) d\tau \quad (2)$$

where $h_j(t)$ is the IRF in the j th direction. By substituting Eq. 1 into 2:

$$\begin{aligned} \chi(t) &= \sum_{j=1}^m \int_0^t \sum_{i=1}^n u_j^{(i)} s_j^{(i)}(\tau) h_j(t - \tau) d\tau \\ &= \sum_{j=1}^m \sum_{i=1}^n u_j^{(i)} a_j^{(i)}(t) = \sum_{j=1}^m \mathbf{a}_j(t) \mathbf{u}_j \end{aligned} \quad (3)$$

where $a_j^{(i)}(t) = \int_0^t s_j^{(i)}(\tau) h_j(t - \tau) d\tau$ and $\mathbf{a}_j(t) = [a_j^{(1)}(t), a_j^{(2)}(t), \dots, a_j^{(n)}(t)]$. With the definition $\mathbf{a}(t) =$



$[a_1(t), a_2(t), \dots, a_m(t)]$, the above equation can be written as $\chi(t) = \mathbf{a}(t) \cdot \mathbf{u}$ and this means that for linear systems the response can be stated as the product of two vectors which one of them is deterministic time variant and the other is random time invariant, if the excitation is stated as the product of two vectors like Eq. 1.

The limit state surface for a linear structure with considering the response threshold X at time point t_n can be written as $G(\mathbf{u}) = X - \chi(t_n) = X - \mathbf{a}(t_n)\mathbf{u}$ which is a hyper-plane in the $m \times n$ dimensional space of normal random variables.

For nonlinear systems Eq. 2 and 3 are not valid but the nonlinear limit state surface $G(\mathbf{u}) = X - \chi(t_n)$ can be approximated by a hyper-plane at the nearest point to the origin \mathbf{u}^* of the normal random variable space in TELM. For finding \mathbf{u}^* which is called design point in reliability, a constrained optimization problem should be solved in a standard normal space with dimensions equal to the elements of \mathbf{u} vector. Design point \mathbf{u}^* is a point on the limit state surface, $G(\mathbf{u}^*) = 0$, with minimum distance from the origin. For obtaining \mathbf{u}^* the response of the structure and its gradient should be calculated and used in a proper optimization framework.

After solving the optimization problem, the non-Gaussian response is replaced by a Gaussian one which is defined by the based function vector $\mathbf{a}(t_n) = \nabla_{\mathbf{u}} \chi(\mathbf{u}, t_n)|_{\mathbf{u}=\mathbf{u}^*(X, t_n)}$ and the probability of failure could be expressed by $\Phi(-\beta(X, t_n))$, where β is called reliability index and is equal to Euclidean norm of \mathbf{u}^* , where $\Phi(\cdot)$ is the standard normal CDF. Based on the geometric properties, $\mathbf{a}(t_n)$ vector can be obtained from the following equation (Fujimura and Der Kiureghian 2007):

$$\mathbf{a}(t_n) = \frac{X}{\|\mathbf{u}^*(X, t_n)\|} \frac{\mathbf{u}^*(X, t_n)}{\|\mathbf{u}^*(X, t_n)\|} \quad (4)$$

The obtained vector from the above equation separated to \mathbf{m} vectors \mathbf{a}_1 to \mathbf{a}_m each with n elements. Then the IRFs of TELS can be obtained from the following equations for $j = 1, \dots, m$.

$$\sum_{k=1}^n h_j(t_n - t_k) s_j^{(i)}(t_k) \Delta t \cong a_j^{(i)}(t_n); \quad i = 1, \dots, n \quad (5)$$

Each of the above relations represent a set of n equations which can be solved for the values of the IRFs at time points. The obtained IRFs indicate TELS for the specified threshold X and time point t_n and define a linear system in the space of \mathbf{u} variables which has an identical design point with the nonlinear system.

By obtaining the IRFs or frequency response functions (FRFs) (by the Fourier transform of IRFs) of equivalent linear system, linear random vibration methods can be used to determine the considered statistical responses for the nonlinear system with first order approximation.

Application of the TELM for bi-axial materials and bi-directional excitation with incident angle θ

Finding the design point is the most computational part in applying TELM. This point is the solution of a constrained optimization problem, which requires calculating the response and its gradient with respect to random variables for several times. Due to the large number of standard normal random variables in stochastic dynamic, this gradient computation should be done by direct differentiation method (DDM) algorithm. Thus to apply TELM to different material models DDM algorithm for those models should be developed. In a recent work the authors have developed bi-axial Bouc–Wen DDM computation for use in TELM (Raoofi and Ghafory-Ashtiany 2013a). TELS and the statistics of the response for a 3D structure with rigid diaphragm supported by four columns with bi-axial Bouc–Wen material subjected to independent white noise and modulated filtered white noise along the structural axes have been obtained, and showed good agreement in comparison with simulation results. But in general the statistically independent components of excitation are not along the structural axes (Penzien and Watabe 1975; Singh and Ghafory-Ashtiany 1984). Thus the authors present TELM for considering incident angle of independent components of excitation in Ghafory-Ashtiany and Raoofi (2013). In the latter case the obtained IRFs are along the principal directions of excitation (these are orthogonal directions where the components of excitation can be stated as uncorrelated and statistically independent components along them) not along structural axes. In other words, if p and q are perpendicular directions which relate to the principal axes of excitation, subscript j in the Eqs. 1 and 6 is $j = p, q$ and the uncertainty vector of the problem is $\mathbf{u} = [\mathbf{u}_p, \mathbf{u}_q]^T$. The following example shows the capabilities of the method.

Example 1 Here the statistical analysis of the nonlinear response of the structural model shown Fig. 1 with the given dynamic properties and Bouc–Wen nonlinear material have been presented. A brief representation on uni-axial and bi-axial nonlinear Bouc–Wen material models has been presented in “Appendix A”. The column properties are selected differently to produce a 3D structure with torsional coupling even for linear case. Mass roof is $m = 1 \text{ KN s}^2/\text{m}$. The desired response is displacement of column C in x direction, i.e. $\chi = d_{Cx}$. The structure subjected to bi-directional white noise excitation with spectral intensity $1 \text{ m}^2/\text{s}^3$ and $0.5 \text{ m}^2/\text{s}^3$ in p and q directions, respectively, with duration $t_n = 10 \text{ s}$. The mean square response of the linear system is equal to $\sigma_0 = 0.129 \text{ m}$.

Figure 2 shows the FRFs of TELSs for incident angles of excitation, $\theta = 0$ and $\theta = 30$, in p and q directions



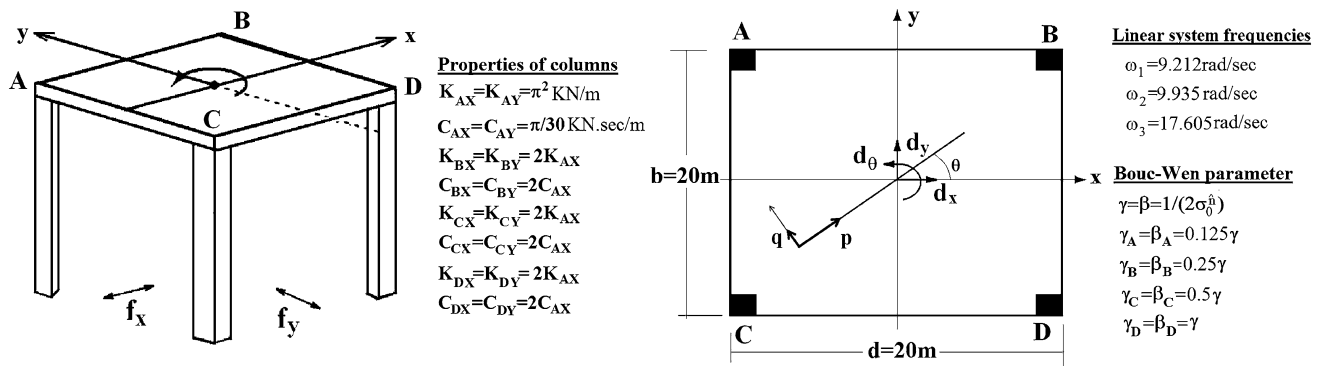


Fig. 1 Structural model, degrees of freedom and input excitations

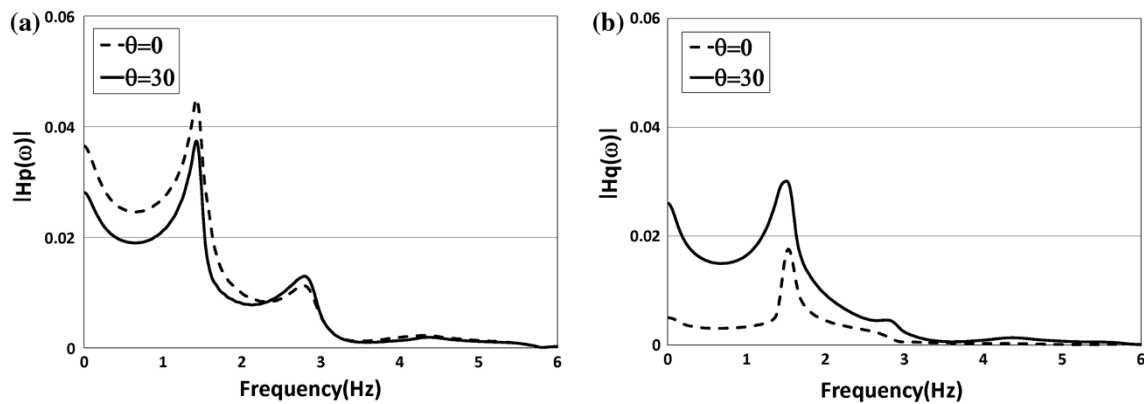


Fig. 2 FRF for response (d_{Cx}) threshold level $4\sigma_0$ for nonlinear system ($\alpha = 0.1, \hat{n} = 2$), excitation is WN excitation with spectral intensity $S_p = 1 \text{ m}^2/\text{s}^3$ and $S_q = 0.5 \text{ m}^2/\text{s}^3$. **a** p direction, **b** q direction

(principal axes of excitation) for response threshold level $4\sigma_0$. As stated in the “Appendix A” parameter \hat{n} defines degree of bi-axial interaction for Bouc–Wen bi-axial model. This parameter is set equal to 2 in this example. For $\theta = 0$ principal axes of earthquake coincide with the major axes of the structure (x and y), thus $h_p = h_x$ and $h_q = h_y$. Since the desired response is in the x direction, the contribution of $h_q = h_y$ (perpendicular direction of the desired response) is related to the asymmetric behavior of the system and bi-axial interaction of the material. For $\theta = 30^\circ$ these FRFs are in the alignment with the principal axes of excitation, and both of them have contribution in the response even though the system would be symmetric or without bi-axial interaction, because none of them is perpendicular to the desired response. These FRFs can be used in the equivalent linear random vibration analysis of the system which is subjected to the independent bi-directional excitation in the p and q directions.

To understand the effects of Bouc–Wen bi-axial interaction parameter \hat{n} on the TELS, FRFs in x and y directions ($\theta = 0$) are obtained for nonlinear systems ($\alpha = 0.1$) with $\hat{n} = 1$ (high bi-axial interaction, rhombus yield surface),

$\hat{n} = 2$ (circular yield surface) and $\hat{n} = 7$ (small bi-axial interaction) for response (d_{Cx}) threshold level $3\sigma_0$ at $t_n = 10 \text{ s}$. The comparisons of the results with the linear system are shown in Fig. 4 for the white noise bi-directional excitation along the structural axes with spectral intensities of $S_x = 1 \text{ m}^2/\text{s}^3$ and $S_y = 0.5 \text{ m}^2/\text{s}^3$ in x and y directions, respectively.

Figure 3 shows that for the three nonlinear cases in comparison with the linear case, the dominant peak in the FRFs becomes smaller and the low frequency portion of the FRFs is amplified. But with increasing \hat{n} parameter, the difference between linear and nonlinear cases has increased in the direction of the desired response (x direction) and decreased in the perpendicular direction. The reason of this phenomenon is that for small bi-axial interaction the dissipation of energy under design point excitation is done almost in one direction which is the direction of the desired response, thus the nonlinearity effects in this direction are higher than strong bi-axial interaction case and are lower in the perpendicular direction.

The complementary CDF, $\Phi(-\beta(X, t_n))$ for nonlinear case with $\alpha = 0.1$ and $\hat{n} = 2$, has been obtained for 20



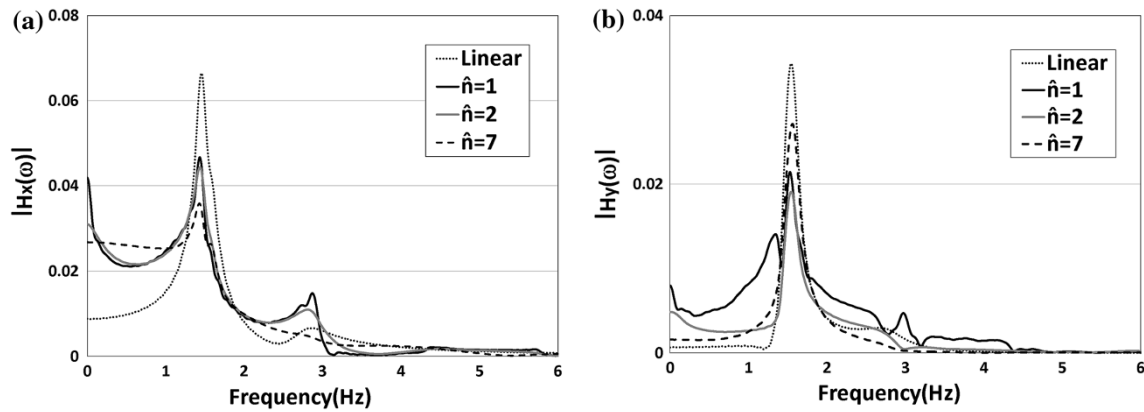


Fig. 3 FRF for linear and nonlinear ($\alpha = 0.1$) system for response d_{Cx} to WN excitation with different yielding surface (\hat{n}) and threshold level of $3\sigma_0$ and $t_n = 10$ s for WN excitation. **a** x , **b** y direction

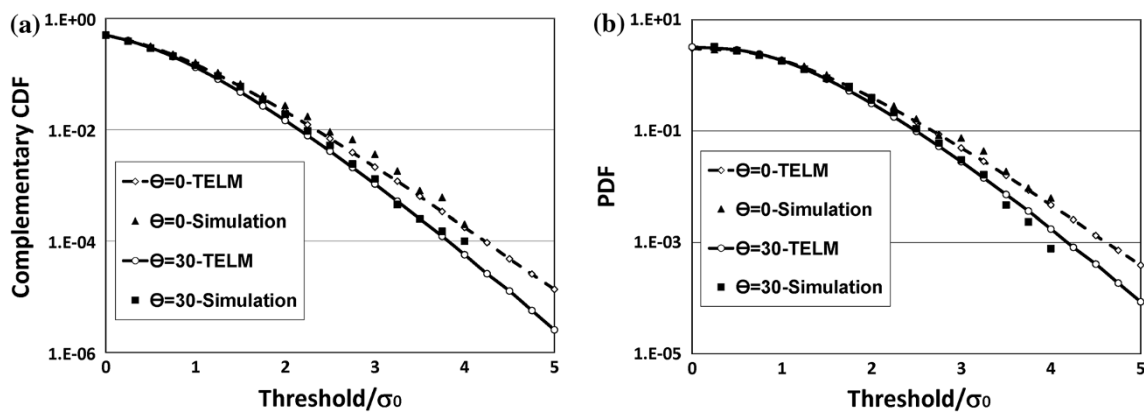


Fig. 4 Complementary CDF (**a**) PDF (**b**) for response d_{Cx} and highly nonlinear system ($\alpha = 0.1$), excitation is WN excitation with spectral intensity $S_p = 1 \text{ m}^2/\text{s}^3$ and $S_q = 0.5 \text{ m}^2/\text{s}^3$ and angle of incidence $\theta = 0$ and $\theta = 30$; TELM and simulation

response threshold levels from $0.25\sigma_0$ to $5\sigma_0$ with intervals $0.25\sigma_0$ and incident angle $\theta = 0$ and $\theta = 30$; and the results are shown versus threshold values X in Fig. 4a.

The probability density functions of TELS at each threshold which can be obtained by $\phi(-\beta(X, t_n))/\|(a(X, t_n))\|$ are shown in Fig. 4b for linear and nonlinear systems. These results are compared with the result of Monte Carlo with 20,000 simulations and show good agreement. The difference between probability values for incident angle $\theta = 0$ and $\theta = 30$ with increasing threshold and, therefore, intensity of nonlinear behavior is evident. Furthermore, it is seen that the probability of failure for $\theta = 0$ is higher than for $\theta = 30$ in all thresholds.

Application of TELM for degrading materials

In a new research the authors have presented modified TELM method which can consider degradation parameters with some more computational effort for obtaining the average rate of hysteretic energy of structure subjected to

random realization of excitations (Raoufi and Ghafory-Ashtiany 2013b). The degradation parameters are functions of severity and cyclic behavior of loading and are dependent to hysteretic energy. Furthermore, these parameters are time dependent and the response is non-stationary even if the excitation is stationary. The results of applying TELM in the usual manner to degrading structures have no agreement with simulations. By considering the dependency of degradation parameters to hysteretic energy and a proper predefining of this energy based on the average rate of hysteretic energy of structure subjected to random realization of excitations the TELMs are used in a modified manner for the uni-axial Bouc–Wen stiffness and strength degrading material model (Raoufi and Ghafory-Ashtiany 2013b). In the continuation of this section a SDOF system with degrading Bouc–Wen material subjected to WN excitation has been considered as a numerical example.

Example 2 A SDOF system subjected to WN base acceleration ($\ddot{d}_g(t)$) with system properties given in Fig. 5 has been considered for the analysis. The force–



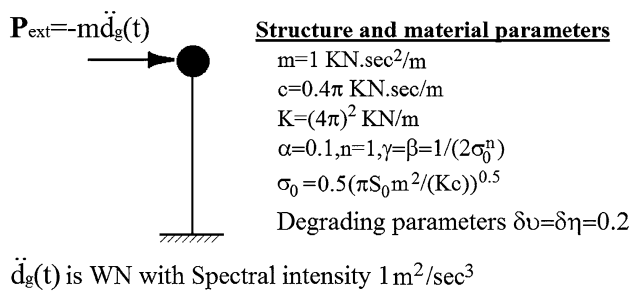
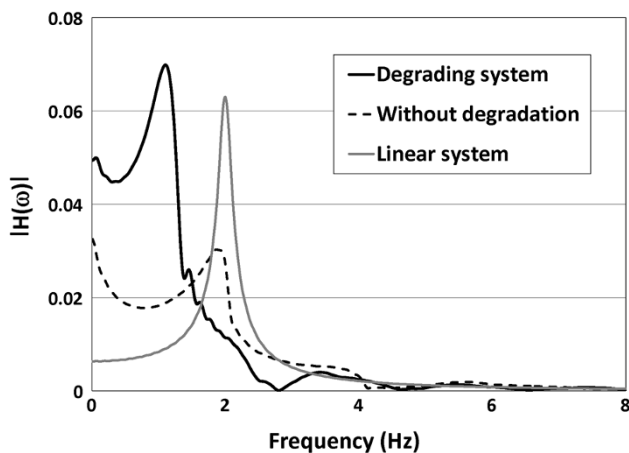


Fig. 5 SDOF structural model

Fig. 6 FRF of degrading system with modified TELM, nonlinear system without degradation for threshold $3\sigma_0$ and linear system

displacement relation of the system is governed by the degrading uni-axial Bouc–Wen model (“Appendix A”). The duration of excitation selected as $t_n = 8 \text{ s}$.

Figure 6 shows the FRF of the degrading system which obtained by modified TELM in comparison with FRF of nonlinear system with no-degradation for threshold $3\sigma_0$ and the linear system. The dominant peak of FRF of modified

TELS moves toward lower frequencies relative to non-degrading system. Furthermore, the values of the FRF increase in the low frequency region and decrease in the high frequency region relative to the non-degrading system.

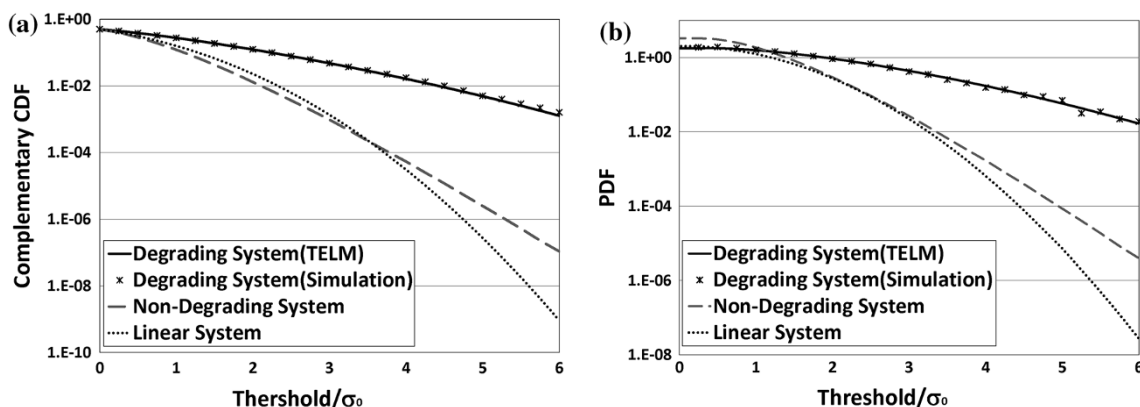
Figure 7a, b shows the complementary CDF and PDF for the degrading system, non-degrading and linear systems. Results of degrading system are compared with the results of Monte Carlo with 10^5 simulations and show good agreement.

It is evident from these figures that the probability of exceeding the response from a specified threshold for degrading system with mentioned specification is very larger than system without degradation and the linear one. Thus considering these phenomena in reliability analysis and performance-based design of structures can be very important.

A new proposed method for simulating rotational component of earthquake excitation and its application in TELM

TELM method has been developed for unidirectional and multi-directional excitations with independent components. For earthquake excitation, it is possible to find principal axes of earthquake which along them the translational components of excitations are uncorrelated and statistically independent. But the rotational component of earthquake usually is stated in terms of translational components and is dependent to these components.

Rezaeian and Der Kirureghian (2011) have presented a method for simulating the independent translational components of earthquake in terms of standard normal random variables to be used in performance-based design and specially in TELM. But there was nothing mentioned about simulating the rotational component of earthquake in terms of standard normal random variables.

Fig. 7 a Complementary CDF, b PDF for linear, non-degrading nonlinear and degrading system ($\delta_v = \delta_\eta = 0.2$) with TELM and simulation

In this paper the presented method in Newmark (1969) has been utilized to describe the rotational component which is based on a simple representation of the ground motion components as traveling waves. In this representation, the rotational components are related to the jerks of the translational components and the shear wave velocity of propagation (Ghafoory-Ashtiany and Singh 1986). Thus if the independent translational acceleration components which are along the structural axes in x and y directions are stated as \ddot{d}_{gx} and \ddot{d}_{gy} , respectively, the rotational acceleration component of earthquake $\ddot{d}_{g\theta}$ can be:

$$\ddot{d}_{g\theta} = \frac{1}{2v_c} \frac{d}{dt} [\ddot{d}_{gx} - \ddot{d}_{gy}] \quad (6)$$

where v_c is the identical shear wave velocity in all directions and d/dt shows differentiation with respect to time. For a three-degree freedom system like rigid one-story roof diaphragm with two translational DOFs in x and y directions and one rotational DOF around z axis, perpendicular to xy surface (Fig. 8) subjected to base acceleration, external excitation vector can be stated as the following:

$$\mathbf{P}_{\text{ext}} = \begin{Bmatrix} f_x \\ f_y \\ f_\theta \end{Bmatrix} = - \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_0 \end{bmatrix} \begin{Bmatrix} \ddot{d}_{gx} \\ \ddot{d}_{gy} \\ \ddot{d}_{g\theta} \end{Bmatrix} \quad (7)$$

where f_x, f_y, f_θ are external forces along DOFs and m and I_0 are the roof mass and inertia of the roof around mass center. Referring to Eq. 1, \ddot{d}_{gx} and \ddot{d}_{gy} can be stated as the following:

$$\ddot{d}_{gx} = -f_x(t)/m = -(1/m) \sum_{i=1}^n u_x^{(i)} s_x^{(i)} = -(1/m) \cdot \mathbf{s}_x(t)^T \mathbf{u}_x \quad (8a)$$

$$\ddot{d}_{gy} = -f_y(t)/m = -(1/m) \sum_{i=1}^n u_y^{(i)} s_y^{(i)} = -(1/m) \cdot \mathbf{s}_y(t)^T \mathbf{u}_y \quad (8b)$$

To define $\ddot{d}_{g\theta}$ in terms of basis time dependent deterministic vector $\mathbf{s} = \{s_x(t) \ s_y(t)\}$ and time invariant random variable vector $\mathbf{u} = \{u_x \ u_y\}$, the Eq. 6 for small time steps Δt will be approximated as:

$$\ddot{d}_{g\theta}(t) = \frac{1}{2v_c} \left[\frac{\ddot{d}_{gx}(t) - \ddot{d}_{gx}(t - \Delta t)}{\Delta t} - \frac{\ddot{d}_{gy}(t) - \ddot{d}_{gy}(t - \Delta t)}{\Delta t} \right] \quad (9)$$

Substituting Eq. 8 into 9 results:

$$\ddot{d}_{g\theta} = (-1/2mv_c\Delta t) [\mathbf{s}_x(t)^T \mathbf{u}_x - \mathbf{s}_x(t - \Delta t)^T \mathbf{u}_x - \mathbf{s}_y(t)^T \mathbf{u}_y + \mathbf{s}_y(t - \Delta t)^T \mathbf{u}_y] \quad (10)$$

With defining $s_{x\theta}(t) = (I_0/2mv_c\Delta t)[s_x(t) - s_x(t - \Delta t)]$ and $s_{y\theta}(t) = (I_0/2mv_c\Delta t)[s_y(t) - s_y(t - \Delta t)]$ the rotational component of earthquake excitation f_θ can be stated as:

$$f_\theta(t) = [\mathbf{s}_{x\theta}(t)^T \mathbf{u}_x - \mathbf{s}_{y\theta}(t)^T \mathbf{u}_y] = \sum_{i=1}^n \left(u_x^{(i)} s_{x\theta}^{(i)} - u_y^{(i)} s_{y\theta}^{(i)} \right) \quad (11)$$

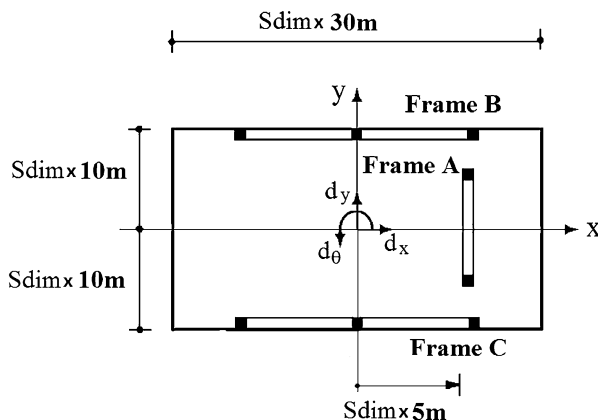
The desired response of a linear structure with considering the rotational component of excitation can be stated as:

$$\chi(t) = \int_0^t f_x(\tau) h_x(t - \tau) d\tau + \int_0^t f_y(\tau) h_y(t - \tau) d\tau + \int_0^t f_\theta(\tau) h_\theta(t - \tau) d\tau \quad (12)$$

Substituting Eqs. 1 and 11 into the latter equation and separating coefficients of $u_x^{(i)}$ and $u_y^{(i)}$ results:

$$\chi(t) = \sum_{i=1}^n a_x^{(i)}(t) u_x^{(i)} + \sum_{i=1}^n a_y^{(i)}(t) u_y^{(i)} = \mathbf{a}_x^T \mathbf{u}_x + \mathbf{a}_y^T \mathbf{u}_y = \mathbf{a}^T \cdot \mathbf{u} \quad (13)$$

where $\mathbf{u} = \{u_x \ u_y\}$, $\mathbf{a} = \{a_x \ a_y\}$ and



Structure and nonlinear material parameters

$$\begin{aligned} M &= 1 \text{ KN} \cdot \text{sec}^2/\text{m} \\ c &= 1.5 \text{ KN} \cdot \text{sec}/\text{m} \\ K &= 280 \text{ KN}/\text{m} \\ \alpha &= 0.5, n=1, \gamma=\beta=1/(2\sigma_0^n) \\ \sigma_0 &= 0.5(\pi S_0 m^2/(Kc))^{0.5} \end{aligned}$$

Natural Frequencies of linear system ($\alpha=1$)

$$f_1 = 2.4070 \text{ Hz}, f_2 = 3.7663 \text{ Hz}, f_3 = 4.0037 \text{ Hz}$$

Fig. 8 Structural model; rigid diaphragm supported by three frames with negligible out-of plan stiffness



$$a_x^{(i)}(t) = \int_0^t \left\{ s_x^{(i)}(\tau) h_x(t-\tau) + s_{x\theta}^{(i)}(\tau) h_\theta(t-\tau) \right\} d\tau \quad (14a)$$

$$a_y^{(i)}(t) = \int_0^t \left\{ s_y^{(i)}(\tau) h_y(t-\tau) - s_{y\theta}^{(i)}(\tau) h_\theta(t-\tau) \right\} d\tau \quad (14b)$$

For nonlinear problems again we can solve the optimization problem and find the design point \mathbf{u}^* . After finding design point excitation the gradient vector of the linearized limit state surface can be obtained from Eq. 4. The obtained vector separated to 2 vectors \mathbf{a}_x and \mathbf{a}_y each with n elements. The result of approximating Eqs. 14a and 14b integrals with simple rectangular rule is as:

$$\sum_{j=1}^n s_x^{(i)}(t_j) h_{x\theta}^{(i)}(t_n - t_j) \Delta t = a_x^{(i)}(t) \quad i = 1, \dots, n \quad (15a)$$

$$\sum_{j=1}^n s_y^{(i)}(t_j) h_{y\theta}^{(i)}(t_n - t_j) \Delta t = a_y^{(i)}(t) \quad i = 1, \dots, n \quad (15b)$$

where

$$h_{x\theta}^{(i)}(t_n - t_j) = h_x(t_n - t_j) + h_\theta(t_n - t_j) \left(s_{x\theta}^{(i)}(t_j) / s_x^{(i)}(t_j) \right) \quad (16a)$$

$$h_{y\theta}^{(i)}(t_n - t_j) = h_y(t_n - t_j) - h_\theta(t_n - t_j) \left(s_{y\theta}^{(i)}(t_j) / s_y^{(i)}(t_j) \right) \quad (16b)$$

If the external force was filtered white noise excitation in the two directions because the presence of $s_{x\theta}^{(i)}(t_j) / s_x^{(i)}(t_j)$ and $s_{y\theta}^{(i)}(t_j) / s_y^{(i)}(t_j)$ terms which are dependent to i , solving Eq. 15 for obtaining $h_{x\theta}^{(i)}(t_n - t_j)$ and $h_{y\theta}^{(i)}(t_n - t_j)$ is difficult or maybe impossible. But for white noise excitations if S_x and S_y are spectral intensity in x and y directions, respectively, the values of elements of vectors $\mathbf{s}_y(t = i \times \Delta t)$ and $\mathbf{s}_y(t = i \times \Delta t)$ are zero, except the i th elements of these vectors which are equal to $\sigma_x = \sqrt{2\pi S_x \Delta t}$ and $\sigma_y = \sqrt{2\pi S_y \Delta t}$, respectively. Thus the horizontal and rotational components at time $t = i \times \Delta t$ i.e. the i th elements of load vector are as the following:

$$f_x(t) = m\sigma_x u_x^{(i)} \quad (17a)$$

$$f_y(t) = m\sigma_y u_y^{(i)} \quad (17b)$$

$$f_\theta(t) = m\sigma_{x\theta} \left(u_x^{(i)} - u_x^{(i-1)} \right) - m\sigma_{y\theta} \left(u_y^{(i)} - u_y^{(i-1)} \right) \quad (17c)$$

where $\sigma_{x\theta} = (I_0/2v_c \Delta t) \sigma_x$ and $\sigma_{y\theta} = (I_0/2v_c \Delta t) \sigma_y$. After substituting Eq. 17a, 17b, 17c into 14a, 14b and using rectangular rule for Duhamel's integral and with a little simplification we have:

$$\chi(t) = \sum_{i=1}^n m\sigma_x h_{x\theta}^{(i)} u_x^{(i)} \Delta t + \sum_{i=1}^n m\sigma_y h_{y\theta}^{(i)} u_y^{(i)} \Delta t \quad (18)$$

where $h_{x\theta}^{(i)} = h_x^{(i)} + (I_0/2mv_c \Delta t) (h_\theta^{(i)} - h_\theta^{(i+1)})$ and $h_{y\theta}^{(i)} = h_y^{(i)} - (I_0/2mv_c \Delta t) (h_\theta^{(i)} - h_\theta^{(i+1)})$, $h_x^{(i)} = h_x(t - t_i)$, $h_y^{(i)} = h_y(t - t_i)$ and $h_\theta^{(i)} = h_\theta(t - t_i)$. Thus the values of $h_{x\theta}^{(i)}$ and $h_{y\theta}^{(i)}$ could be obtained as follows:

$$h_{x\theta}^{(i)} = a_x^{(i)} / m\sigma_x \Delta t; \quad h_{y\theta}^{(i)} = a_y^{(i)} / m\sigma_y \Delta t \quad (19)$$

Therefore, the equivalent linear system is defined by two IRFs which are affected by the presence of dependent rotational component.

It is worthy of note that the presented method for simulating the dependent rotational component is a rough proposal method. To obtain a more precise method it is required to compare the simulated component with real database of earthquake and match the results. In the continuation of this paper to investigate the application of the proposed method in TELM analysis a numerical example has been presented.

Example 3 A3D structure with a rigid roof diaphragm which is supported by two frames in x and one frame in y direction as shown in Fig. 8 is considered. The out-of-plane stiffness and damping of frames are negligible. The in-plane internal force of frames is stated by non-degrading uni-axial Bouc–Wen model (“Appendix A”). The initial stiffness, K , damping, c , and Bouc–Wen properties of all frames are identical and have been shown in Fig. 8. The frames are massless and the mass of the rigid diaphragm M and natural frequencies of system are also shown in this Figure.

Excitation is independent bi-directional white noise base acceleration with identical spectral intensity in x and y directions with $S_0 = 1 \text{ m}^2/\text{s}^3$. Furthermore the duration of the excitation is set as $t_n = 6 \text{ s}$.

The dimensions of roof diaphragm are $b = S_{\text{dim}} \times 30 \text{ m}$, $d = S_{\text{dim}} \times 20 \text{ m}$ and the eccentricities of the frame A is $e = S_{\text{dim}} \times 5 \text{ m}$, where S_{dim} is the scale parameter for considering systems with different dimension without changing in the natural frequencies of the system. The desired response is displacement of frame C in the x direction.

Even though the frequency content of excitations is dependent on the soil type and shear wave velocity, but in this study only wide band white noise translational excitation without considering the effects of soil properties is used for extending TELM with dependent rotational component.



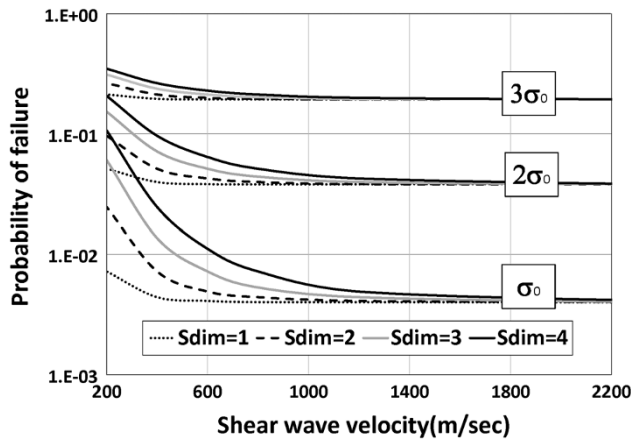


Fig. 9 Effects of shear wave velocity and structural dimension on the probability of failure for different response (d_{Cx}) threshold level for WN excitation with spectral intensity $S_x = S_y = 1 \text{ m}^2/\text{s}^3$

Figure 9 shows the variations of failure probability, exceeding the desired response, d_{Cx} , from specified thresholds ($\sigma_0, 2\sigma_0, 3\sigma_0$) at specified time instant $t_n = 6\text{ s}$ for the nonlinear system with different scale of dimensions (S_{dim}) and shear wave velocities. It can be seen in this figure for high shear wave velocities and for different thresholds increasing the dimension of structure has a little effect on the failure probabilities. But with decreasing the shear wave velocity the failure probabilities for different S_{dim} will be very different. In other words based on the definition for rotational component of earthquake the importance of considering this component for larger structures in lower shear wave velocities will be increased.

Figure 10 shows the FRFs for linear system ($\alpha = 1$) with considering rotational component for $S_{\text{dim}} = 1$ and $S_{\text{dim}} = 4$ for 200 m/s shear wave velocity, and FRFs without considering rotational component (WRC). These FRFs can be obtained by TELM or by modal analysis of the system. The results of modal analysis of the linear system have been shown in “Appendix B”. In the case WRC, FRF

in x direction is equal to FRF of the second mode. When the rotational component is considered the effects of the first and the third modes which relate to H_θ in Eq. 14a, 14b would appear. This effect is higher for larger S_{dim} . H_y is related to the first and the third modes in all three cases in Fig. 10b, but the effect of the third mode increases by increasing the effect of the rotational component. The effect of rotational component for $S_{\text{dim}} = 1$ is not considerable even though the shear wave velocity is very small.

Figure 11 shows the FRFs for the nonlinear system ($\alpha = 0.5$) and response (d_{Cx}) threshold level $3\sigma_0$ for the case WRC and two case with considering the rotational component related to $S_{\text{dim}} = 1$ and $S_{\text{dim}} = 4$. This Figure shows that same as the linear case with considering the effect of rotational component the FRFs in the two directions for the two latter cases will increase around the frequencies related to the first and third modes. The effect of considering rotational component will increase with increasing the dimension of the structure.

The CDF, $\Phi(-\beta(X, t_n))$, has been obtained for different response threshold levels for nonlinear system ($\alpha = 0.5$) and the case without rotational component and two cases with considering this component with $S_{\text{dim}} = 1$ and $S_{\text{dim}} = 4$ for shear wave velocity 200 m/s; and the results are shown versus threshold values X in Fig. 12a.

The PDF of TELS would be obtained by $\phi(-\beta(X, t_n))/(a(X, t_n))$. The obtained PDF for the all three cases are shown in Fig. 12b. The comparison of the results with the results of Monte Carlo with 20,000 simulations shows good agreement.

Summary and conclusions

TELM is an equivalent linearization method which uses the advantages of the first order reliability method. In this method stochastic excitation is discretized and represented

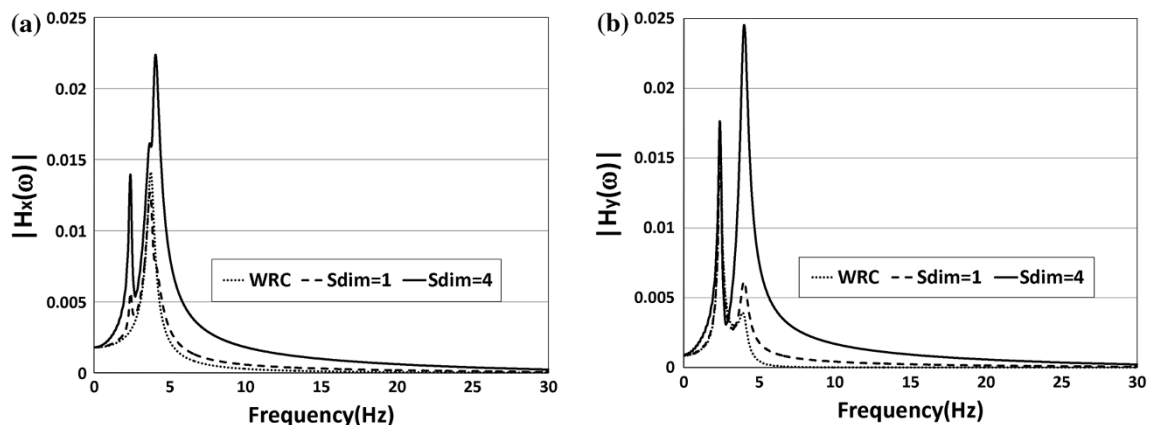


Fig. 10 FRF of linear system for response (d_{Cx}), excitation is white noise with spectral intensity $S_x = S_y = 1 \text{ m}^2/\text{s}^3$ without considering rotational component (WRC) and with considering this component with different scales. **a** x direction, **b** y direction



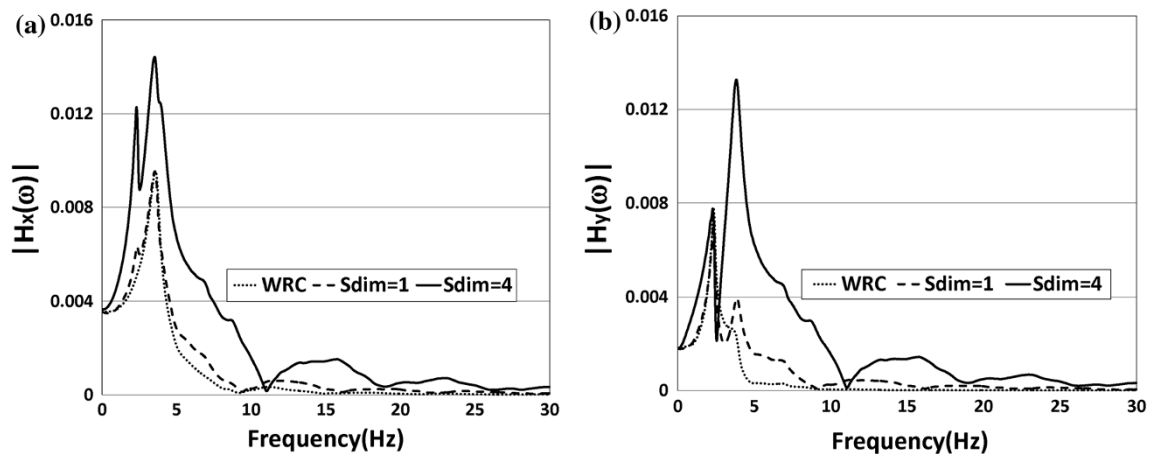


Fig. 11 FRF for nonlinear ($\alpha = 0.5$) system for response (d_{Cx}) threshold level $3\sigma_0$ to WN excitation with spectral intensity $S_x = S_y = 1 \text{ m}^2/\text{s}^3$, without considering rotational component (WRC) and with considering this component with different scales. **a** x direction, **b** y direction

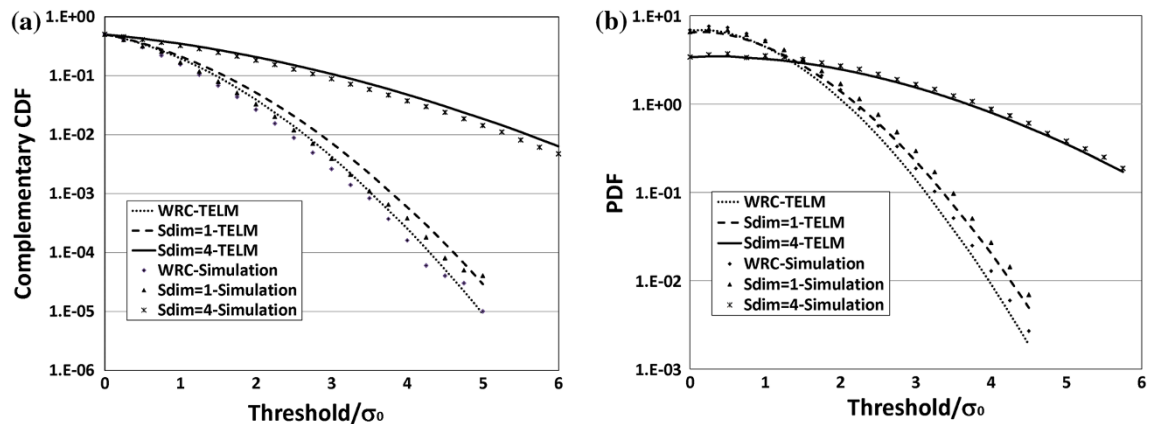


Fig. 12 Complementary CDF (a) PDF (b) for nonlinear system ($\alpha = 0.5$) without considering rotational component (WRC) and with considering this component with different scales with TELM and simulation

in terms of a finite set of standard normal random variables. In the space of these normal random variables the nonlinear limit state surface for a specified threshold and time will be linearized at design point which is the nearest point to the origin of the standard normal space. This linearization is accomplished by unit impulse response functions (IRFs) for nonlinear system for each direction of excitation. These IRFs can be used for obtaining the statistical properties of nonlinear system by linear random vibration methods. This method applied to non-degrading material for two-dimensional single- and multi-degrees of freedom structures and three-dimensional one-story structure supported by in-plane uni-axial stiffness frames which is subjected to independent random excitation along the structural axes and shows good results in comparison with conventional equivalent linearization method and simulation in previous works. But to use TELM for more practical problems it is required to extend this method to cover more realistic material and excitation characteristics.

In this regard the method has been extended for 3D structures subjected to bi-directional excitation with different incident angle with bi-axial behavior of material. In addition, the method with a little modification has been applied to structures with degrading materials. The comprehensive description and required algorithms and equations for these developments have been presented in the previous works of the authors. In this paper a brief review and two numerical analyses have been presented.

Furthermore in this paper a new approach has been proposed for simulating rotational component of earthquake excitation in terms of translational independent components and TELM has been applied for a 3D structure with considering rotational component of earthquake. Numerical example shows the abilities of TELM in predicting the probabilities of failure in comparison with simulation results.

The proposed method for simulating dependent rotational component in terms of standard normal variables



should be checked and calibrated with real rotational component of earthquake in future works.

More investigations are needed to further generalize TELM; some of these have been listed below:

- In finding the design point it is required to find the response and its sensitivity with direct differentiation method, this is the most challenging part of TELM, thus developing the sensitivity analysis for using the other material models is necessary.
- TELM in present and previous works have been only applied to 2D stick-like models or shear beam models or 3D models with rigid diaphragm supported by nonlinear columns or frames. It is required to develop this method for more general nonlinear systems including frames with nonlinear beam and columns and using models capable of predicting the dominant yield pattern under random excitations.
- Developing of TELM for degrading material has been done only for uni-axial Bouc–Wen model, it is required to consider this behavior for bi-axial materials too.
- In the all previous works the excitation had been white noise or modulated filtered white noise, using simulated real earthquakes as inputs would be a good subject for research.
- The considered response in all the previous works had been displacement response, investigating about other response quantities such as forces, moments and application of damage indexes would be interesting.
- Finally application of the TELM in performance-based assessment and design framework would be the subject of future investigations.

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Appendix A: Bouc–Wen material model

Uni-axial Bouc–Wen model

Bouc–Wen class models have been widely used to efficiently describe smooth hysteretic behavior in time history and random vibration analyses (Song and Der Kiureghian 2006). This model which in the first time presented for modeling uni-axial non-degrading hysteretic behavior by Bouc (1967) was generalized and used in nonlinear random

vibration in Wen (1976), this model by adding new parameters developed for considering stiffness and strength degradation in Baber and Wen (1981). Even though this model was developed for considering pinching behavior of materials in Baber and Noori (1986) the model of Baber and Wen (1981) has been considered in this paper and pinching behavior has been eliminated.

The nonlinear restoring force in Bouc–Wen model can be stated as the following for each nonlinear elements of the model:

$$P_{\text{int}}(t) = \alpha K d(t) + (1 - \alpha) K z(t) \quad (20)$$

where P_{int} is the nonlinear hysteretic force, d is the relative displacement of nonlinear element, K is the initial stiffness, α is the ratio of the yielding stiffness to the initial stiffness and z is the hysteretic component of displacement:

$$\dot{z} = (1/\eta) \left\{ \dot{d} - v \left(\beta |\dot{d}| |z|^{\hat{n}-1} z + \gamma \dot{d} |z|^{\hat{n}} \right) \right\} \quad (21)$$

In the above relation dot shows the derivative with respect to time. γ , β and \hat{n} are parameters that control the basic hysteretic shape, v and η are, respectively, strength and stiffness degradation shape functions and define as the following:

$$v(\varepsilon) = 1 + \delta_v \varepsilon \quad (22)$$

$$\eta(\varepsilon) = 1 + \delta_\eta \varepsilon \quad (23)$$

where δ_v and δ_η , respectively, determine the rate of strength and stiffness degradations. Strength and stiffness degradation are controlled by the hysteretic energy dissipation ε that be defined as:

$$\varepsilon = (1 - \alpha) K \int_0^t z \dot{u} dt \quad (24)$$

For non-degrading case δ_v and δ_η are zeros.

Bi-axial Bouc–Wen

In the bi-axial Bouc–Wen model (Park et al. 1986; Wen and Yeh 1989) restoring force in x and y directions is as follows:

$$\{P_{\text{int}x}; P_{\text{int}y}\}^T = \alpha \{K_x d_x; K_y d_y\}^T + (1 - \alpha) \{K_x z_x; K_y z_y\}^T \quad (25)$$

where K_x and K_y are initial stiffness coefficients; z_x and z_y are hysteretic components of displacement of element, respectively, in x and y directions and α is the ratio of stiffness after yielding to the initial stiffness. Hysteretic components of displacement can be stated as the following:



$$\begin{aligned}
\dot{z}_x &= \dot{d}_x - z_x |\dot{d}_x| |z_x|^{\hat{n}-1} [\beta + \gamma \operatorname{sgn}(\dot{d}_x z_x)] \\
&\quad - z_x |\dot{d}_y| |z_y|^{\hat{n}-1} [\beta + \gamma \operatorname{sgn}(\dot{d}_y z_y)] \\
\dot{z}_y &= \dot{d}_y - z_y |\dot{d}_x| |z_x|^{\hat{n}-1} [\beta + \gamma \operatorname{sgn}(\dot{d}_x z_x)] \\
&\quad - z_y |\dot{d}_y| |z_y|^{\hat{n}-1} [\beta + \gamma \operatorname{sgn}(\dot{d}_y z_y)]
\end{aligned} \quad (26)$$

where $\operatorname{sgn}(\cdot)$ is the sign function.

γ and β are constant parameters of the model and control the shape of hysteresis loops. Furthermore, \hat{n} is a natural number parameter that represents the yield surface or rate of bi-axial interaction. If $\hat{n} = 1$ the yield surface becomes a rhombus and implies very significant interaction and with increasing \hat{n} the effect of interaction decreases. For $\hat{n} = 2$ the yield surface for isotropic case is a circle that represents equal yield capacity along any direction and with further increasing as n approaches infinity, the yield surface becomes a square and the yield strength along one axis is independent of the displacement along its orthogonal axis (Lee and Hong 2010).

Appendix B: Relation between the modal IRFs and the IRFs in the directions of the excitations

With considering degrees of freedom as d_x , d_y and d_θ and $S_{\text{dim}} = 1$ mass, \mathbf{M} , stiffness, \mathbf{K} , and damping, \mathbf{C} , matrices are as the following:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1300/12 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 560 & 0 & 0 \\ 0 & 280 & 1400 \\ 0 & 1400 & 63000 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1.5 & 7.5 \\ 0 & 7.5 & 337.5 \end{bmatrix}$$

Eigenvalue and eigenvector matrices of this system are:

$$\mathbf{\Phi} = \begin{bmatrix} 0 & 1 & 0 \\ -0.9344 & 0 & 0.3562 \\ 0.0342 & 0 & 0.0898 \end{bmatrix},$$

$$\omega^2 = \begin{bmatrix} 228.7206 & 0 & 0 \\ 0 & 560 & 0 \\ 0 & 0 & 632.8179 \end{bmatrix}$$

where ω^2 and $\mathbf{\Phi}$ are eigenvalue and eigenvector matrices. For the desired response $\chi = d_{cx} = d_x + (d/2)d_\theta$ the relation between impulse response function in the direction of DOFs and modal IRFs, h_1 , h_2 , and h_3 are as follows:

$$\begin{aligned}
h_x &= h_2(t_n - \tau) \\
h_y &= \{-0.3195648h_1(t_n - \tau) + 0.3198676h_3(t_n - \tau)\} \\
h_\theta &= \{0.0116964h_1(t_n - \tau) + 0.0806404h_3(t_n - \tau)\}
\end{aligned}$$

To obtain $h_{x\theta}$ and $h_{y\theta}$ it is required to obtain the above relations into Eq. 16a, 16b.

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